An Adaptive Meshfree Strong-form Method Based on Regularized Least-squares Formulation

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Summary

An adaptive meshfree method based on strong-form and regularized least-squares formulation is proposed. Since strong-form method is used, it possesses many attractive features such as simplicity in formulation which eases the implementation of adaptive scheme. To solve the instability issue encountered in a strong-form method so as to use it in the adaptive analyses, a regularized least-squares procedure is employed. As the stability is restored, the present meshfree strong-form method with adaptive capability is successfully implemented and tested using a number of examples including solving Poisson's equation and solid mechanics problems.

Introduction

In the past few decades, meshfree methods that formulated based on locally supported nodes have been actively studied and achieved remarkable success. According to the formulation procedure, meshfree methods can be classified into three major categories: meshfree strong-form method, meshfree weak-form method and meshfree weak-strong-form method [1]. Among these three major categories, strong-form method has the simplest formulation procedure, which can ease the implementation of adaptive schemes. However, strong-form method encounters instability problems, which on other hand makes the implementation of adaptive scheme impossible. Without an effective stabilization measure, it is not practical to use strong-form method for adaptive analyses.

In this paper, a regularized least-squares procedure is employed to stabilize the solutions of the radial point collocation method (RPCM) that uses radial basis functions (RBFs) and locally supported nodes for field function approximation [1]. As stability is restored, the regularized least-squares RPCM or RLS-RPCM is then incorporated with an error estimation and refinement procedure for adaptive analyses. The regularized least-squares procedure not only successfully restores the stability but also makes the stiffness matrix to possess some good properties such as symmetry and positive definite (SPD), which offsets the additional costs in the computation.

Error indicator based on the interpolation error [5] is adopted in our adaptive RLS-RPCM. This indicator is simple but yet effectively reflects the quality of the local approximation. The Voronoi diagram is used to locate the position of additional nodes to be inserted in refinement process.

Function Approximation

Consider a field function $u(\mathbf{x})$ in a problem domain Ω. The value of the field function at point of interest $\mathbf{x}$ can be approximated using shape functions. These shape functions are created through a simple interpolation procedure using basis functions and n supporting nodes in the local support domain. The local field function approximation can then be expressed using shape functions as

$$u^h(\mathbf{x}) = \mathbf{\Phi}^T(\mathbf{x})\mathbf{U}_s \tag{1}$$

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where $\mathbf{U}_s$ is a vector of unknown nodal field variables at each node in the local support domain

$$\mathbf{U}_s{}^T = \{u_1 \quad u_2 \quad \cdots \quad u_n\} \tag{2}$$

and $\mathbf{\Phi}^T(\mathbf{x})$ is a vector of shape functions which has the form of

$$\mathbf{\Phi}^T(\mathbf{x}) = \{\phi_1 \quad \phi_2 \quad \cdots \quad \phi_n\} \tag{3}$$

in which $\phi_i \;(i = 1, 2\cdots,\ n)$ is the RPIM shape function for node i in the local support domain. The details of the formulation can be found in the references [e.g., 1, 2]. In this work, Multi-quadrics radial basis functions (MQ-RBFs) augmented with completed 2nd order polynomial functions are used as basis functions to create shape functions.

Radial Point Collocation Method

Radial point collocation method (RPCM) is a strong-form meshfree method that used RPIM shape functions for approximation based on locally supported nodes through simple collocation procedure (see e.g., [1, 6]).

Assuming that the governing PDEs defined in the problem domain Ω can be described as

$$A(u) = 0 \qquad \text{in } \Omega \tag{4}$$

with Neumann boundary conditions

$$B(u) = 0 \qquad \text{on } \Gamma_t \tag{5}$$

and Dirichlet boundary conditions

$$u = \bar{u}_i \qquad \text{on } \Gamma_u \tag{6}$$

where $A(\;)$ and $B(\;)$ are the differential operators.

Using equation (1), the discretized system equations can be formed by collocating: 1) equation (4) at all internal nodes; 2) equation (5) at the nodes on the Neumann boundary; and 3) equation (6) at the nodes on the Dirichlet boundary. A set of resultant algebraic equations can be assembled in the matrix form as follows.

$$\mathbf{KU} = \mathbf{F} \tag{7}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{U}$ is the vector of unknown nodal field variables and $\mathbf{F}$ is the nodal force vector.

Regularized Least-Squares Procedure

In this work, a regularized least-squares procedure that is used in solving inverse problems (see, e.g., [3]) is employed here for our forward problem to stabilize the solutions of RPCM. The procedure of regularization is described as follows.

First, a functional Π is defined in the form of

$$\Pi = \{\mathbf{KU} - \mathbf{F}\}^T \{\mathbf{KU} - \mathbf{F}\} + \alpha \{\mathbf{RU} - \mathbf{T}\}^T \{\mathbf{RU} - \mathbf{T}\} \tag{8}$$

where α is a regularization factor which determines the degree of regularization, **R** is the regularization matrix and **T** is the regularization nodal force vector.

To form the regularization matrix $\mathbf{R}$ and regularization nodal force vector $\mathbf{T}$, additional prior information for the system is required. In this work, we create $\mathbf{R}$ and $\mathbf{T}$ in the following manner. It is well known that Neumann boundary condition often leads to instability (see, e.g., [1]). The used of additional conditions on the Neumann boundary can improve the stability as discussed in e.g., [1]. Therefore, we impose also the governing PDEs in the collocation form at the nodes only on the Neumann boundary.

$$A\left(u_i^{\,h}\right) = 0 \qquad on \quad \Gamma_t \tag{9}$$

A set of resultant algebraic equations can then be obtained in the matrix form as

$$\mathbf{RU} - \mathbf{T} = 0 \tag{10}$$

In seeking the minima of the functional Π with respect to $\mathbf{U}$, we have

$$\partial \Pi / \partial \mathbf{U} = 2\mathbf{K}^T \{\mathbf{KU} - \mathbf{F}\} + 2\alpha \mathbf{R}^T \{\mathbf{RU} - \mathbf{T}\} = 0 \tag{11}$$

which gives

$$\left[\mathbf{K}^T\mathbf{K} + \alpha \mathbf{R}^T\mathbf{R}\right]\mathbf{U} = \mathbf{K}^T\mathbf{F} + \alpha \mathbf{R}^T\mathbf{T} \tag{12}$$

or

$$\hat{\mathbf{K}}\mathbf{U} = \hat{\mathbf{F}} \tag{13}$$

where $\hat{\mathbf{K}} = \left[\mathbf{K}^T\mathbf{K} + \alpha \mathbf{R}^T\mathbf{R}\right]$ is the regularized stiffness matrix and $\hat{\mathbf{F}} = \mathbf{K}^T\mathbf{F} + \alpha \mathbf{R}^T\mathbf{T}$ is the regularized nodal force vector.

Finally, the vector of unknown nodal field variables $\mathbf{U}$ can be obtained by

$$\mathbf{U} = \hat{\mathbf{K}}^{-1}\hat{\mathbf{F}} \tag{14}$$

if $\hat{\mathbf{K}}$ is not singular matrix and well conditioned.

Form equation (12) we can observe that the regularized stiffness matrix $\hat{\mathbf{K}}$ is symmetric and hence at least non-negative definite. The additional $\alpha \mathbf{R}^T\mathbf{R}$ increase the positivity of the stiffness matrix and hence $\hat{\mathbf{K}}$ is often positive definite. Note that if the regularization factor $\alpha = 0$, the RLS-RPCM reduces to the RPCM.

It is very crucial to determine an appropriate value of regularization factor α because the accuracy of the solutions depends on the regularization factor used. On one hand, a large regularization factor can provide more stability; on the other hand, we do not want to lose too much on accuracy. The guideline of selecting an appropriate regularization factor is to use the smallest regularization factor that is just enough to restore the stability. In this work, the L-curve method [4] is used to determine the appropriate value of regularization factor. Note that α only needs to be determined at the initial step and used in all the consequent adaptive steps, as we found that α is insensitive to the number of the nodes used.

Adaptive Scheme

In this work, error indicator proposed in [5] is adopted here and defined as follows.

$$\eta(\mathbf{x}_i) = \left|u^s(\mathbf{x}_i) - u^{\bar{s}}(\mathbf{x}_i)\right| \tag{15}$$

where $u^s(\mathbf{x}_i)$ is the value of field function at node *i* and $u^{\bar{s}}(\mathbf{x}_i)$ is the reference value of field function at node *i*. These values are evaluated by interpolating the values of the field function at the nodes in local support domain of node *i*. S is the nodal set in the support domain of node *i* and $\bar{S} = S \setminus \{\mathbf{x}_i\}$.

This error indicator reflects the quality of local reproduction of the interpolation. The refinement process will be executed if the following criteria are met.

$$\eta(\mathbf{x}_i) > \kappa_1 \eta^* \quad and \quad \eta^* > \kappa_2 u_{\max} \tag{16}$$

where κ_1 and κ_2 are the tolerant values which $0 < \kappa_i < 1$; η^* is the maximum value of error indicator in the entire problem domain; $u_{\max}$ is the maximum value of the field function in the entire problem domain.

In this work, the Voronoi diagram is used to locate the position for the additional nodes to be inserted in the refinement process.

Numerical Examples

A Poisson's equation and a solid mechanics problem are solved using the RLS-RPCM to examine and demonstrate the proposed adaptive method. In the computation, the shape parameters in MQ-RBF are chosen as $\alpha_c = 3.0$ and $q = 1.03$. Tolerant values $\kappa_1 = \kappa_2 = 0.05$ are used in the adaptive scheme. The following error norm is used for the purpose of examining the accuracy of the results of the present adaptive meshfree RLS-RPCM.

$$e = \sqrt{\left(\sum \left(u^{exact} - u^{appro}\right)^2 \Big/ \sum \left(u^{exact}\right)^2\right)} \tag{17}$$

Example 1:

Consider a Poisson's equation defined in $\Omega = [0,1] \times [0,1]$

$$\nabla^2 u = \sin \pi x \sin \pi y \qquad \text{in } \Omega \tag{18}$$

The Neumann boundary conditions are

$$\partial u / \partial x = -1/2\pi \ \cos \pi x \sin \pi y \qquad \text{along } x = 0 \text{ and } x = 1 \tag{19}$$

$$\partial u / \partial y = -1/2\pi \ \sin \pi x \cos \pi y \qquad \text{along } y = 1 \tag{20}$$

and the Dirichlet boundary condition is

$$u = -1/2\pi^2 \ \sin \pi x \sin \pi y \qquad \text{along } y = 0 \tag{21}$$

In this example, 15 nodes in the local support domain are used for creating RPIM shape functions. Regularization factor $\alpha = 0.5$ is determined at the initial step. The adaptive analysis starts from 25 regularly distributed nodes initially and stops at 5^{th} step of iteration with 199 nodes as shown in Figure 1. As the exact field function is a smooth function, the nodes are regularly distributed in the domain, see Figure 1.

The error norm of *u* is drastically reduced from 47.52% to 0.29% through five adaptive steps as shown in Figure 2. The *u* along $x = 0.5$ at initial step and last step are plotted in Figure 3 for comparison.

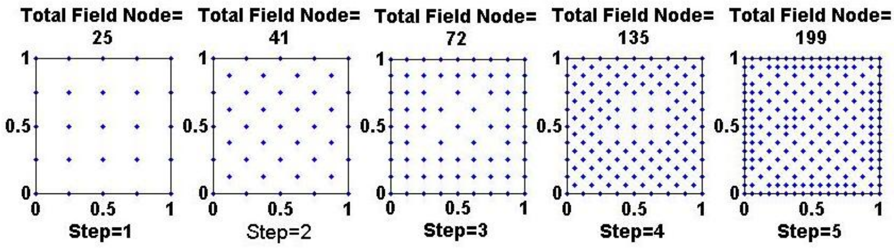

Figure 1. Node distributions for Poisson's equation problem at each adaptive step

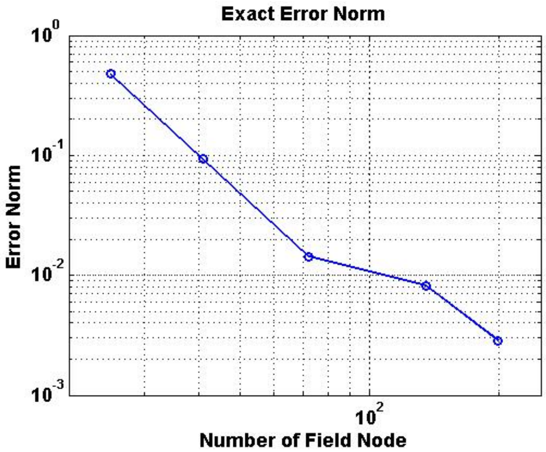

Figure 2. Error norm of u at each adaptive step for Poisson equation's problem

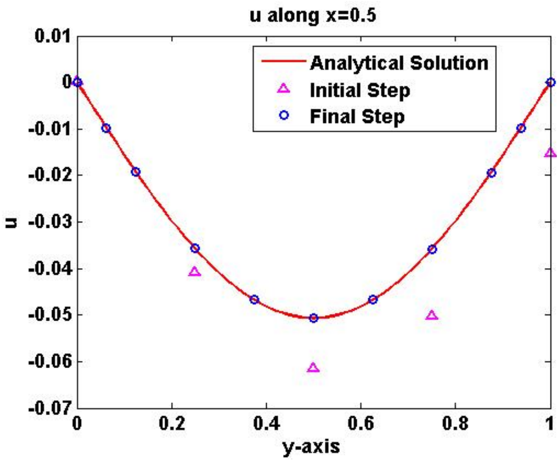

Figure 3. The u along $x = 0.5$ at the initial step and final step

Example 2:

In the second example, an infinite plate with circular hole subjected to an uniaxial traction in horizontal direction is studied and it is considered as a plain strain problem. The geometries and material properties are taken as $a = 0.2m$, $b = 2.0m$, Young's modulus $E = 1 \times 10^3\ N/m^2$, Poisson's ratio $\upsilon = 0.3$. As this problem is symmetric, only quarter of the problem domain is modelled.

The governing PDE is described as

$$\sigma_{ij,j} + b_i = 0 \qquad \text{in } \Omega \tag{22}$$

with Neumann boundary conditions

$$\sigma_{ij} n_j = t_i \qquad \text{on } \Gamma_t \tag{23}$$

and Dirichlet boundary conditions

$$u_i = \bar{u}_i \qquad \text{on } \Gamma_u \tag{24}$$

In this example, 30 nodes in the local support domain are used for creating RPIM shape functions. Regularization factor $\alpha = 0.005$ is determined at the initial step. The adaptive analysis starts from 145 nodes at initial step and terminates at 3rd step with 507 nodes.

The error norm of Von Mises stress is drastically reduced from 46.44% to 5.4% as shown in Figure 5. The normal stresses σ_{xx} along the left edge of the plate at initial and final steps are plotted in Figure 6 for comparison.

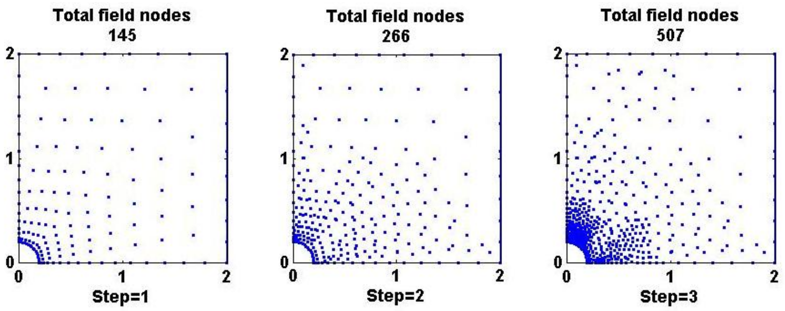

Figure 4. Node distribution of infinite plate with circular hole at each adaptive step

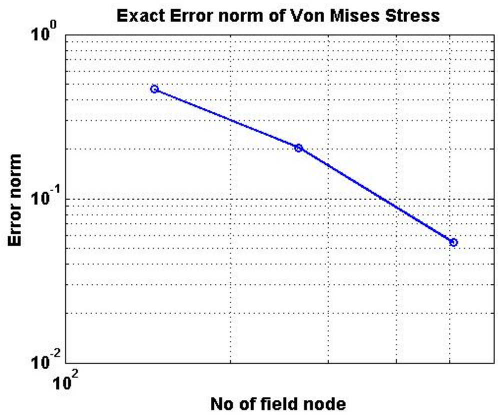

Figure 5. Error norm of Von Mises stress at each adaptive step

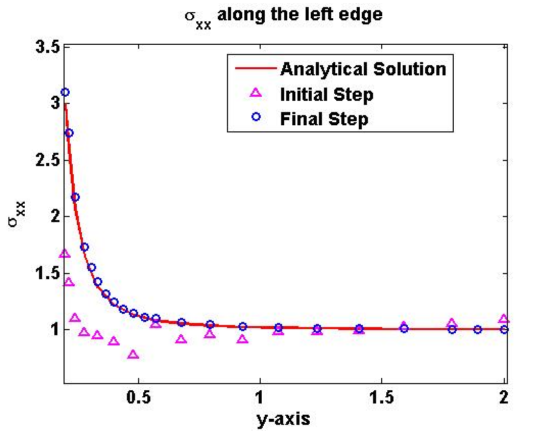

Figure 6. The normal stress σ_{xx} along the left edge of plate at initial and final steps

Conclusion

In this work, a regularized least-squares procedure that is used in solving inverse problems is successfully employed to overcome the instability issue in RPCM for solving forward problems governed by PDEs. As stability is restored, RLS-RPCM can then be used in the adaptive analyses. The numerical examples have shown that RLS-RPCM is very easy to implement as an adaptive method to achieve stable results of desired accuracy.

Reference

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